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A multi-factor stochastic model and estimation procedure for the valuation and hedging of commodity contingent claims

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ABSTRACT

In this paper we provide a new multi-factor stochastic model of commodity futures prices and propose a Kalman filter estimation procedure that may be applied to a panel data with missing observations. This model may be used to implement financial engineering applications which require the valuation or hedging of commodity-linked assets. We calibrate our model using daily oil futures prices and analyze its goodness-of-fit to observed prices and empirical volatilities.

1. INTRODUCTION

The valuation and hedging of commodity contingent claims has received a great amount of attention by both academics and practitioners, and has become an important area of financial engineering. Inextricably interwoven with this issue is the modelling and estimation of the stochastic behaviour of commodity prices. The practical implications of having better models and estimation methodologies is that commodity producers and consumers, and also financial intermediaries, may implement sound investment and risk-management strategies with vast economic implications.

One important source of information for the study of commodity prices is the futures market. It is known that the futures price of any financial asset should be equal to the spot price plus the cost-of-carry. The cost-of-carry represents the storage cost plus the interest paid to finance the asset minus the net benefit that accrues to the holder of the asset, if any. In the commodities literature, the benefit received by the owner of the commodity, but not by the owner of a futures contract, is called the convenience yield [Brennan (1991)], which is commonly represented as a dividend yield.

Several models of the stochastic process followed by commodity prices have been proposed in the literature. They basically differ in how they specify spot price innovations and the cost-of-carry. Early models of commodity prices assumed a one-factor geometric Brownian motion for the spot price with a constant cost-of-carry. [Brennan and Schwartz (1985)]. Even though this widely used and simple model has the advantage of being very tractable, it has some undesirable properties, like exhibiting constant-volatility term-structure of futures prices. Empirical evidence suggests, however, that the volatility term-structure of futures prices is a decreasing function of maturity [Bessembinder et. al. (1996)], which may be explained by the existence of mean reversion in commodity prices [Bessembinder et. al. (1995)].

To address this issue, several authors have proposed different one-factor models that take into account mean reversion in commodity prices [Laughton and Jacobi (1993 and 1995), Ross (1995), Schwartz (1997), Cortazar and Schwartz (1997)]. However, an empirical implication of all models that consider a single source of uncertainty is that futures prices for different maturities should be perfectly correlated, which defies existing evidence.

To account for a more realistic model of commodity prices, two-factor and three-factor models have been proposed [Gibson and Schwartz (1990), Schwartz (1997), Schwartz and Smith (2000), Schwartz (1997), Cortazar and Schwartz (2003)]. The advantage of using more factors in modelling the spot price process is that a better fit to observed prices may be obtained. This goodness-of-fit can generally be observed not only in terms of mean square errors, but also by comparing empirical and model-implied volatility term-structures which is critical for risk management applications. [Cortazar and Schwartz (2003)].

In addition to defining the price model, the methodology to obtain parameter estimates must be chosen. Some difficulties must be addressed, however, to successfully apply these models to commodity markets. For example, most multi-factor models are based on non-observable state variables that must be estimated from observed prices. Also, for any given date the number of available prices is generally higher than the number of state variables that need to be estimated. Thus we must assume that observed prices have measurement errors which have to be assigned across the different contracts.

To estimate these dynamic models, several econometric procedures have been proposed in the literature. One of the most successful procedures is the Kalman filter, a widely used estimation methodology which can handle multi-factor models with non-observable state variables and measurements errors. Also, it is able to use a large panel-data of prices in the estimation process. The Kalman filter has been used in finance to estimate commodity price models by Schwartz (1997), and Schwartz and Smith (2000), among others.

One of the problems of traditional implementations of the Kalman filter in finance applications, is that it normally assumes a complete panel-data set. This implies that for all given dates in the estimation sample, prices for the same set of contracts (with the same maturities) must be observed. This is not normally the case, because financial markets have innovations and new contracts with longer maturities are frequently introduced. Traditional applications of the Kalman filter typically address this missing-data problem by aggregating or discarding data with the consequent loss of information [Schwartz (1997)].

The missing-data problem may be so relevant that some authors have chosen not to use the Kalman filter but to propose an alternative procedure to handle cases where the panel-data is incomplete. Cortazar and Schwartz (2003) propose a very simple estimation procedure and apply it to a panel of oil futures prices. The methodology, however, does not make an optimal use of the price information (as opposed to the optimal Kalman filter) and is unable to obtain estimation errors of parameters. Others, have proposed adjusting the Kalman filter procedure to address incomplete panel-data conditions. Sorensen (2002) uses this procedure for a seasonal price model and Cortazar, Schwartz and Naranjo (2003) for estimating the term structure of interest rates in an emerging market.

In this paper we propose a new multi-factor model of commodity prices which generalizes existing three-factor models [Schwartz (1997), Cortazar and Schwartz (2003)] to an N-factor setting. We also show how to estimate the model using the Kalman filter in an incomplete panel-data setting, making an optimal use of all market prices available. We then apply a four-factor model to an oil futures prices data set which includes all oil futures contracts traded at NYMEX during 10 years. We show that our model has an excellent fit to the data, with low mean square errors and good adjustment to the empirical volatility term structure which is crucial for risk management purposes.

The organization of the paper is the following. The next section explains the new N-factor model. In section 3 we present the standard Kalman filter methodology and how to extend it in an incomplete panel-data setting. Section 4 presents estimation results for oil futures and section 5 finally concludes.

2. A GENERALIZED LOG-NORMAL MODEL OF COMMODITY PRICES

In this section we describe a new general N-factor lognormal model for the spot price of a commodity, which generalizes existing 3-factor models found in the literature [Schwartz (1997), Cortazar and Schwartz (2003)].

Even though one-factor models may be able to explain a sizable fraction of total price variance, these models tend to fit rather poorly the volatility term structure, making them inadequate to support an effective risk management strategy. For many risk management and financial engineering applications it is important to use a price model that replicates very closely the empirical volatility term structure, which implies using models with several factors.

The optimal number of factors that should be specified in a model depends on the stochastic behaviour of the prices of the specific commodity that is being modelled and on the complexity that the modeller is willing to accept. One way to analyse the behaviour of prices is to make a PCA analysis and find the fraction of the explained variance that a model with different number of factors may provide. For example, Cortazar and Schwartz (1994) applied this procedure to explain the dynamics of copper futures prices and found that models with 1, 2 or 3 factors could explain around 90%, 98% and 99% of the variance, respectively. Depending on the model used, it may be important to specify a commodity price model with a high number of factors.

The N-factor model presented in this paper extends existing models of commodity prices to an arbitrary number of factors while providing simple analytic valuation formulas for futures prices, which renders the model tractable and easy to implement and calibrate.

In this model, we consider that the log-spot price process $\log S_t$ of the commodity can be described as:

$$\log S_t = \mathbf{1}'\mathbf{x}_t + \mu t \tag{1}$$

where $\mathbf{x}_t$ is a $n \times 1$ vector of state variables and μ is a constant. The vector of state variables $\mathbf{x}_t$ is governed by the following stochastic differential equation:

$$d\mathbf{x}_t = -\mathbf{K}\mathbf{x}_t dt + \mathbf{\Sigma} d\mathbf{w}_t \tag{2}$$

where $\mathbf{K} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & \kappa_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \kappa_n \end{pmatrix}$ and $\mathbf{\Sigma} = \begin{pmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_n \end{pmatrix}$ are $n \times n$ diagonal matrices with entries that are positive constants and term-wise different. Also, $d\mathbf{w}_t$ is a $n \times 1$ vector of correlated Brownian motion increments such that:

$$(d\mathbf{w}_t)'(d\mathbf{w}_t) = \mathbf{\Omega} dt \qquad (3)$$

where the (i, j) element of $\mathbf{\Omega}$ is $\rho_{ij} \in [1, -1]$, the instantaneous correlation of state variables i and j.

This model specification implies that the state variables have the Multivariate Normal distribution. The first state variable follows a random walk, and having the stochastic process a unit-root. This means that perturbations in the spot price represented by this state variable are permanent and do not decay with time. Therefore, the process followed by the spot price in this model is non-stationary.

Each of the other state variables reverts to 0, at a mean reversion rate[3] given by k_i. The mean-reverting state variables can be interpreted as non-permanent perturbations in the spot price which do not have long-term effects. These perturbations have an impact on the spot price that decays over time at a rate inversely proportional to its mean reverting parameter κ_i.

Thus, according to equation (1), the spot price has a long term growth rate given by μ. Note that this is a canonical model in the sense that it contains the minimum number of parameters that can be econometrically identified [Dai and Singleton (2001)].

It is important to note that our N-factor model can be easily extended to include seasonality in the commodity price process by adding a seasonal component

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[3] In a mean reverting model, every perturbation is on average reduced by half in $\log(2)/k_i$ units of time.

in the logarithm of the spot price. This would extend the Sørensen (2002) model to an N-factor setting.

By assuming a constant market price of risk[4] $\boldsymbol{\lambda}$, the risk-adjusted process for the vector of the state variables is:

$$d\mathbf{x}_t = -(\boldsymbol{\lambda} + \mathbf{K}\mathbf{x}_t)dt + \boldsymbol{\Sigma} d\mathbf{w}_t \tag{4}$$

where $\boldsymbol{\lambda}$ is a $n \times 1$ vector of constants.

As an example of how our model generalises the existing literature, in appendix A we derive the relationship between our model and Cortazar and Schwartz (2003).

The price of a futures contract at time t and maturing at T can then be found by using traditional no-arbitrage arguments:

$$\begin{aligned} F(\mathbf{x}_t, t, T) = \exp\Bigg(& x_1(t) + \sum_{i=2}^{N} \mathrm{e}^{-\kappa_i (T-t)} x_i(t) + \mu t + (\mu - \lambda_1 + \frac{1}{2}\sigma_1^2)(T-t) \\ & - \sum_{i=2}^{N} \frac{1 - \mathrm{e}^{-\kappa_i (T-t)}}{\kappa_i} \lambda_i + \frac{1}{2} \sum_{i \cdot j \neq 1} \sigma_i \sigma_j \rho_{ij} \frac{1 - e^{-(\kappa_i + \kappa_j)(T-t)}}{\kappa_i + \kappa_j} \Bigg) \end{aligned} \tag{5}$$

where $\kappa_1 = 0$.

This model has the advantage of being tractable enough to obtain simple analytic futures price formulas even for an arbitrary number of factors. Also, the logarithm of the futures price is a linear function of state variables, a fact that will be useful when estimating the model with our Kalman filter-based estimation procedure.

The cost-of-carry $y(\mathbf{x}_t, t, T)$ in this model is given by:

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[4] We assume for simplicity that the market price of risk is constant, but this could be extended to any linear function of the state variables to reflect a possible correlation between the spot price and the cost-of-carry [Casassus and Collin-Dufresne (2001)].

$$y(\mathbf{x}_t, t, T) = -\sum_{i=2}^{N}(1-\mathrm{e}^{-\kappa_i(T-t)})x_i(t) + (\mu - \lambda_1 + \frac{1}{2}\sigma_1^2)(T-t)$$
$$-\sum_{i=2}^{N}\frac{1-\mathrm{e}^{-\kappa_i(T-t)}}{\kappa_i}\lambda_i + \frac{1}{2}\sum_{i,j\neq 1}\sigma_i\sigma_j\rho_{ij}\frac{1-e^{-(\kappa_i+\kappa_j)(T-t)}}{\kappa_i+\kappa_j} \quad (6)$$

It is important to note that the cost-of-carry in this model is a function of only the mean-reverting state variables, implying that the cost-of-carry itself is mean-reverting. Given that the cost-of-carry represents the difference between the interest rate and the convenience yield of the commodity, it can also be assumed that both processes are mean-reverting [Schwartz (1997)][5].

The volatility term structure of futures returns can also be obtained from equation (2) and (5):

$$\sigma_F^2(\tau) = \sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_i\sigma_j\rho_{ij}e^{-(\kappa_i+\kappa_j)\tau} \quad (7)$$

where we assume that $\kappa_1 = 0$.

As the maturity of a futures contract grows, the volatility of futures returns tends to a constant given by σ_1, which is the volatility of the first state variable.

3. KALMAN FILTER-BASED ESTIMATION WITH MISSING OBSERVATIONS

The Kalman filter is an estimation methodology which recursively calculates optimal estimates of unobservable state variables using all past information. Consistent parameter estimates can also be obtained by maximizing the likelihood function of error innovations. In the finance literature, the Kalman filter has been used to estimate and implement stochastic models of commodities

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[5] Note that for the cost-of-carry to be stationary, it is not necessary that both the interest rate and the convenience yield are stationary. For example, they may also be two cointegrating processes with unit-roots. However, we will only focus in modelling of the cost-of-carry.

[Schwartz (1997), Schwartz and Smith (2000), and Sørensen (2002)], interest rates [Lund (1994, 1997), Duan and Simonato (1995), Geyer and Pichler (1998), Babbs and Nowman (1999), de Jong and Santa-Clara (1999), de Jong (2000), Cortazar, Schwartz and Naranjo (2003)], and other relevant economic variables [Pennacchi (1991), and Dewachter and Maes (2001)]. Although widely used in a complete panel-data setting, the literature has not focused on using the Kalman filter where there are missing observations in the panel, a very common feature in many commodity futures markets.

One of the characteristics of the Kalman filter is that estimates of the state variables are obtained using a rich information set that includes past information and not only current prices. The recursive nature of the Kalman filter includes all past data without recurring to extensive calculations. Moreover, it can allow for measurement errors in observable variables, which can be induced by market imperfections, or by the inability of a model with a restricted number of factors to explain the whole structure of contemporaneous observations. Also, the use of panel-data can avoid estimation problems caused by nearly unit-root processes [Ball and Torous (1996)]. Finally, if state variables have a normal distribution, estimates are optimal in a mean square error sense.

In the next subsection we describe the mathematics of the standard Kalman filter and then we explain how to extend it to the case where the panel-data is incomplete.

3.1 Traditional Kalman Filter Estimation

The Kalman filter [Kalman (1960)] is an estimation methodology that initially was used in a wide variety of engineering applications. In this section we give a very general overview of the mathematics behind the Kalman filter. More details can be found in Harvey (1989) or Hamilton (1994).

The Kalman filter may be applied to dynamic models that are in a state-space representation. The measurement equation relates a vector of observable variables $\mathbf{z}_t$ with a vector of state variables $\mathbf{x}_t$:

$$\mathbf{z}_t = \mathbf{H}_t \mathbf{x}_t + \mathbf{d}_t + \mathbf{v}_t \qquad \mathbf{v}_t \sim N(\mathbf{0}, \mathbf{R}_t) \tag{8}$$

where $\mathbf{z}_t$ is a $m \times 1$ vector, $\mathbf{H}_t$ is a $m \times n$ matrix, $\mathbf{x}_t$ is a $n \times 1$ vector, $\mathbf{d}_t$ is a $m \times 1$ vector and $\mathbf{v}_t$ is a $m \times 1$ vector of serially uncorrelated Gaussian disturbances with mean $\mathbf{0}$ and covariance matrix $\mathbf{R}_t$. The assumption that the vector $\mathbf{z}_t$ of observable variables is of a fixed size will be relaxed later to allow for incomplete panel-data.

Measurement equation (8) assumes the existence of a linear relation between observed variables and state variables. As noted above, in our model for the spot price, the logarithm of futures prices is a linear function of state variables. Nevertheless, the Kalman filter could be modified to allow for non-linear measurements equations, as would be the case if, for example, commodity option prices were to be used as observations.

The transition equation describes the stochastic process followed by the state variables:

$$\mathbf{x}_t = \mathbf{A}_t \mathbf{x}_{t-1} + \mathbf{c}_t + \boldsymbol{\varepsilon}_t \qquad \boldsymbol{\varepsilon}_t \sim N(\mathbf{0}, \mathbf{Q}_t) \tag{9}$$

where $\mathbf{A}_t$ is a $n \times n$ matrix, $\mathbf{c}_t$ is an $n \times 1$ vector and $\boldsymbol{\varepsilon}_t$ is an $n \times 1$ vector of serially uncorrelated Gaussian disturbances with mean $\mathbf{0}$ and covariance matrix $\mathbf{Q}_t$, which distribute multivariate Gaussian. The linear relationship established by equation (9) can also be relaxed to include non-gaussian processes for the state variables [Harvey (1989)].

Given the state space representation, the Kalman filter calculates optimal estimates $\hat{\mathbf{x}}_t$ of state variables given all past information. Let $\mathbf{P}_t$ be the variance-covariance matrix of estimation errors of state variables:

$$\mathbf{P}_t = \mathrm{E}(\mathbf{x}_t - \hat{\mathbf{x}}_t)(\mathbf{x}_t - \hat{\mathbf{x}}_t)^T \tag{10}$$

The Kalman filter then works recursively using the previous estimation of the state variables and its associated variance-covariance matrix of the estimation error, given by $\hat{\mathbf{x}}_{t-1}$ and $\mathbf{P}_{t-1}$ respectively.

First, the one-step ahead prediction at time t of the state variables $\hat{\mathbf{x}}_{t|t-1}$ and its error variance-covariance matrix $\mathbf{P}_{t|t-1}$ given all information up to time $t-1$, are computed. This is usually called the prediction step:

$$\hat{\mathbf{x}}_{t|t-1} = \mathbf{A}_t \hat{\mathbf{x}}_{t-1} + \mathbf{c}_t \tag{11}$$

$$\mathbf{P}_{t|t-1} = \mathbf{A}_t \mathbf{P}_{t-1} \mathbf{A}'_t + \mathbf{Q}_t \tag{12}$$

This allows for the calculation of a one-step ahead prediction of the observed variables $\hat{\mathbf{z}}_{t|t-1}$:

$$\hat{\mathbf{z}}_{t|t-1} = \mathbf{H}_t \hat{\mathbf{x}}_{t|t-1} + \mathbf{d}_t \tag{13}$$

When new information $\mathbf{z}_t$ is available, the error of the prediction or innovation $\mathbf{v}_t$, and its associated variance-covariance matrix $\mathbf{F}_t$ are:

$$\mathbf{v}_t = \mathbf{z}_t - \hat{\mathbf{z}}_{t|t-1} \tag{14}$$

$$\mathbf{F}_t = \mathbf{H}_t \mathbf{P}_{t|t-1} \mathbf{H}'_t + \mathbf{R}_t \tag{15}$$

The optimal estimates of the state variables and of the error variance-covariance matrix are then computed in what is called the update step:

$$\hat{\mathbf{x}}_t = \hat{\mathbf{x}}_{t|t-1} + \mathbf{P}_{t|t-1} \mathbf{H}'_t \mathbf{F}_t^{-1} \mathbf{v}_t \tag{16}$$

$$\mathbf{P}_t = \mathbf{P}_{t|t-1} - \mathbf{P}_{t|t-1} \mathbf{H}'_t \mathbf{F}_t^{-1} \mathbf{H}_t \mathbf{P}_{t|t-1} \tag{17}$$

The estimation of the state variables can be interpreted as a Bayesian estimation because the distribution of the error prediction of the observable variables, given all past observations $\{\mathbf{z}_t\}_{t=1}^{t-1}$ and the new information $\mathbf{z_t}$, is known. The Kalman filter estimate of the state variables $\hat{\mathbf{x}}_t$ is then the conditional expectation of $\mathbf{x}_t$, i.e. $\hat{\mathbf{x}}_t = \mathrm{E}_{t-1}(\mathbf{x}_t | \mathbf{z}_t)$. It can be shown [see for example Øksendal (1998)] that this conditional expectation is in fact an optimal estimation, in a mean square error sense, when the stochastic processes followed by the state variables are Gaussian, i.e. when the measurement and transition equations are linear.

The estimation of model parameters $\hat{\boldsymbol{\psi}}$ can be obtained by maximizing the log-likelihood function of error innovations:

$$\log L(\boldsymbol{\psi}) = -\frac{1}{2} \sum_t \log |\mathbf{F}_t| - \frac{1}{2} \sum_t \mathbf{v}_t{}' \mathbf{F}_t^{-1} \mathbf{v}_t \tag{18}$$

where $\boldsymbol{\psi}$ represents a vector containing the unknown parameters. Under normality assumptions, parameters estimates are unbiased and consistent, and distribute asymptotically normal with mean $\mathbf{0}$ and variance-covariance matrix given by $\mathbf{I}(\hat{\boldsymbol{\psi}})^{-1}$, where $\mathbf{I}(\boldsymbol{\psi})$ is the Fisher information matrix:

$$\mathbf{I}(\boldsymbol{\psi}) = \frac{\partial^2 \log L(\boldsymbol{\psi})}{\partial \boldsymbol{\psi} \partial \boldsymbol{\psi}'} \tag{19}$$

There is no close-form solution of the information matrix and second derivatives must be calculated numerically. At the optimum $\hat{\boldsymbol{\psi}}$, and when all parameters are identifiable, the information matrix $\mathbf{I}(\hat{\boldsymbol{\psi}})$ is invertible and definite-positive.

3.2 Kalman Filter Estimation in an incomplete Panel-Data setting

In many commodity markets, not all futures contracts trade every day. Also, financial innovations are common and when new contracts are introduced there is no historical information on them. Therefore, if a complete data set of prices wants to be used, current prices on these new contracts must be discarded, with great information loss.

As noted before a better alternative is to modify the Kalman filter to be used under incomplete data set conditions, which is explained in what follows.

Let $\{\mathbf{z}_t\}_{t=1}^{T_N}$ be a series of vector observations and m_t be the number of observations available at time t, which does not have to be equal to the number of observations available at any other date. This will be called an incomplete panel-data set, where the number of observations available at any date is time dependent. The measurement equation is the same as before:

$$\mathbf{z}_t = \mathbf{H}_t \mathbf{x}_t + \mathbf{d}_t + \mathbf{v}_t \qquad \mathbf{v}_t \sim N(\mathbf{0}, \mathbf{R}_t) \tag{20}$$

but now $\mathbf{z}_t$ is a $m_t \times 1$ vector, $\mathbf{H}_t$ is a $m_t \times n$ matrix, $\mathbf{x}_t$ is a $n \times 1$ vector, $\mathbf{d}_t$ is a $m_t \times 1$ vector and $\mathbf{v}_t$ is a $m_t \times 1$ vector of serially uncorrelated Gaussian disturbances with mean $\mathbf{0}$ and covariance matrix $\mathbf{R}_t$ of dimension $m_t \times m_t$.

The prediction of the state variables $\hat{\mathbf{x}}_{t|t-1}$ and the calculation of the variance-covariance matrix $\mathbf{P}_{t|t-1}$ given previous estimations $\hat{\mathbf{x}}_{t-1}$ and $\mathbf{P}_{t-1}$, are based on equations (11) and (12), which only consider the dynamic properties of state variables and are not affected by the dimension of the vector of observable variables.

When the filter includes the new information $\mathbf{z}_t$, the same equations (16) and (17) can then be used to calculate optimal estimates of the state vector $\hat{\mathbf{x}}_t$ and of the covariance matrix $\mathbf{P}_t$. In fact, what is required is that for each observation $z_{i,t}$, there exists a measurement equation relating this observation with the state variables. For example, if futures contracts are used as observations, then for each maturity there exists a measurement equation that gives the futures price as a function of state variables and its maturity. With this relationship the m_t row vectors of the matrix $\mathbf{H}_t$, and the elements of the vector $\mathbf{d}_t$ can be filled. Also, the variance-covariance matrix of measurement error $\mathbf{R}_t$ must be parameterised. One of the simplest ways to do this is by assuming that all measurements errors are independent and have the same variance, which induces a diagonal variance-covariance matrix $\mathbf{R}_t$ [Babbs and Nowman (1997)].

Another way to look at this is to note that the Kalman filter computes at every date the conditional expectation $\hat{\mathbf{x}}_t = \mathrm{E}_{t-1}(\mathbf{x}_t | \mathbf{z}_t)$. This expectation can be obtained even if the number of observations vary with time because it depends on the statistical relationship between observed and state variables. Of course, the accuracy of the estimation, measured by the variance of the estimation error $\mathbf{P}_t$, increases with the number of observations that are available to update the filter.

4. EMPIRICAL RESULTS FOR THE OIL MARKET

To illustrate and analyse the performance of our commodity price model and estimation procedure, we estimate a 4-factor model using daily light-sweet crude oil futures prices traded at NYMEX.

The oil market is nowadays one of the most important and strategic commodity markets. It can be characterized as a very volatile market, with annualized volatilities of spot price returns reaching almost 40%. Therefore, it is of paramount importance for valuation and hedging purposes to select an accurate model that reflects market dynamics, and also to have reliable and accurate parameter estimates of that model.

Also, oil futures markets have in recent years included new futures contracts with long maturities that currently reach up to 7 years. This makes the oil market very interesting to analyse because long term price behaviour can be extracted from these contracts. For these two reasons we test our model and estimation procedure using this market.

4.1 Description of the Data

The data used consists of all daily light-sweet crude oil futures prices traded at NYMEX during the 10 year period from January 1992 to December 2001.

There are currently 35 contracts traded for different maturities ranging from 1 to 30 months, and 3, 4, 5, 6 and 7 years. However, from 1992 to 1996, the maximum maturity traded at NYMEX was only 4 years. In 1997 new contracts were introduced to include maturities up to 7 years.

To make our analysis we divide the data into three different panels. Panel A includes all futures contracts traded between 1992 and 2001, which corresponds to 70584 observations. Panel B and Panel C include futures contracts traded in the 1992-1996 and 1997-2001 periods, with 30424 and 40160 observations., respectively. Table 1 presents a qualitative description of the data showing the average number of daily observations and the maximum maturity available for each year in the 1992-2001 period.

Table 1: Average number of daily observations and maximum maturity available of light sweet crude oil futures contracts in the period 1992-2001.

<table>
<tr><th></th><th></th><th>Year</th><th>Avg. Number of Daily Observations</th><th>Maximum Maturity (Years)</th></tr>
<tr><td rowspan="10">Panel A</td><td rowspan="5">Panel B</td><td>1992</td><td>22</td><td>3</td></tr>
<tr><td>1993</td><td>22</td><td>3</td></tr>
<tr><td>1994</td><td>21</td><td>3</td></tr>
<tr><td>1995</td><td>25</td><td>4</td></tr>
<tr><td>1996</td><td>31</td><td>4</td></tr>
<tr><td rowspan="5">Panel C</td><td>1997</td><td>34</td><td>7</td></tr>
<tr><td>1998</td><td>31</td><td>7</td></tr>
<tr><td>1999</td><td>31</td><td>7</td></tr>
<tr><td>2000</td><td>33</td><td>7</td></tr>
<tr><td>2001</td><td>34</td><td>7</td></tr>
</table>

Figure 1 characterises the data for the period 1992-2001 showing the average price for each maturity plus and minus one standard deviation. From this figure we can observe that the average price for each maturity ranges approximately from \$19 to \$21. The price standard deviation can be very high for short term contracts, reaching almost 24% of the average price. Long term futures contracts present a lower volatility, which can be 7% of the average futures price.

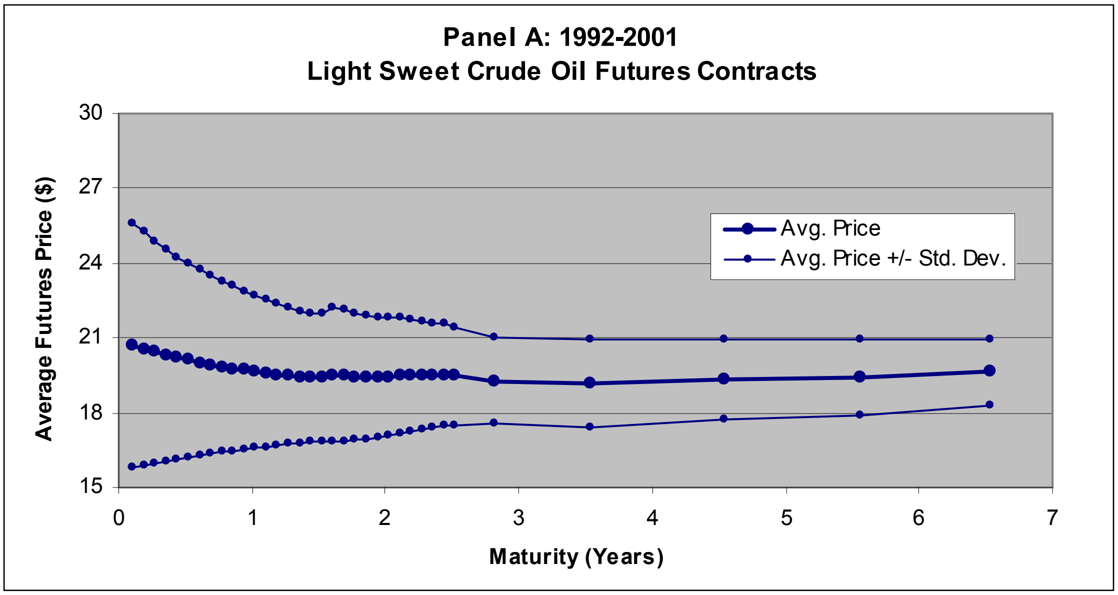

Figure 1: Average price and price standard deviation for light sweet crude oil futures contracts for different maturities in the period 1992-2001.

4.2 Estimation and Model Performance Analysis

We estimate a 4-factor generalized log-normal model using daily observations of light sweet crude oil futures for three different panel-data as indicated in the previous sub-section: Panel A 1992-2001, Panel B 1992-1996 and Panel C 1997-2001. The details of the implementation of the filter can be found in Appendix B.

Parameters estimates are shown in Table 2 with their respective standard error. From this table we can see that most parameters are stable across different panels, which shows the reliability of the model when applied to the oil market. All mean reversion parameters κ_i, for all three panels, are highly significant and show the existence of strong mean reversion in oil prices. Volatility parameters σ_i are also highly significant and stable across panels. Correlation parameters ρ_{ij} are almost all significant, although they account for more standard deviation error than mean reversion and volatility parameters. For each panel, the long term growth rate parameter μ is measured with a large standard deviation error, and is not statistically significant, as almost all market price of risk parameters λ_i, which is similar to what is found in the literature [Schwartz (1997)]. The standard deviation measurement error parameter ξ is small although very significant.

Table 2: Parameter estimations from light sweet crude oil futures data for each panel. Standard errors are shown in parentheses.

	Panel A (1992-2001)	**Panel B (1992-1996)**	**Panel C (1997-2001)**
κ_2	0.415 (0.002)	0.681 (0.013)	0.415 (0.003)
κ_3	1.201 (0.005)	1.283 (0.012)	1.239 (0.010)
κ_4	5.471 (0.039)	7.255 (0.038)	3.545 (0.073)
σ_1	0.191 (0.003)	0.150 (0.003)	0.210 (0.004)
σ_2	0.207 (0.004)	0.230 (0.009)	0.253 (0.007)
σ_3	0.305 (0.007)	0.246 (0.010)	0.334 (0.009)
σ_4	0.260 (0.005)	0.214 (0.005)	0.355 (0.009)
ρ_{21}	-0.336 (0.025)	-0.258 (0.032)	-0.279 (0.025)
ρ_{31}	0.138 (0.029)	0.242 (0.033)	0.070 (0.026)
ρ_{32}	-0.423 (0.025)	-0.714 (0.025)	-0.442 (0.027)
ρ_{41}	-0.010 (0.027)	0.036 (0.035)	0.018 (0.025)
ρ_{42}	0.420 (0.023)	0.348 (0.035)	0.572 (0.028)
ρ_{43}	-0.338 (0.027)	-0.162 (0.040)	-0.554 (0.021)
μ	0.004 (0.059)	-0.007 (0.066)	0.020 (0.093)
λ_1	0.013 (0.059)	-0.019 (0.066)	0.035 (0.093)
λ_2	0.002 (0.054)	-0.012 (0.081)	0.026 (0.080)
λ_3	0.117 (0.089)	0.169 (0.095)	0.185 (0.128)
λ_4	-0.073 (0.079)	0.101 (0.091)	-0.208 (0.143)
ξ	0.003 (0.000)	0.002 (0.000)	0.004 (0.000)

Besides the stability of parameter estimates, we measured the performance of our model and estimation procedure by analysing the fit to the observed futures prices and the empirical volatility term structure.

Figures 2 and 3 shows the fit of the model to 2 different term structures for arbitrary days, when the futures term structure was in contango and backwardation respectively. These 2 days were chosen so as to be representatives of the whole sample period. From these figures it can be seen that the model fits very well to observed prices.

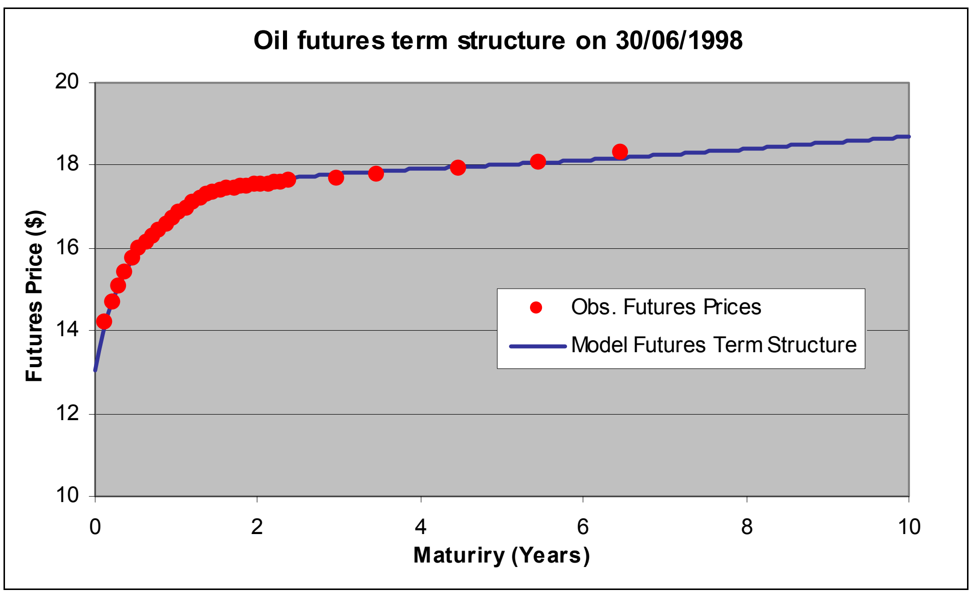

Fig. 2. Estimated and observed oil futures prices on 06/30/1998.

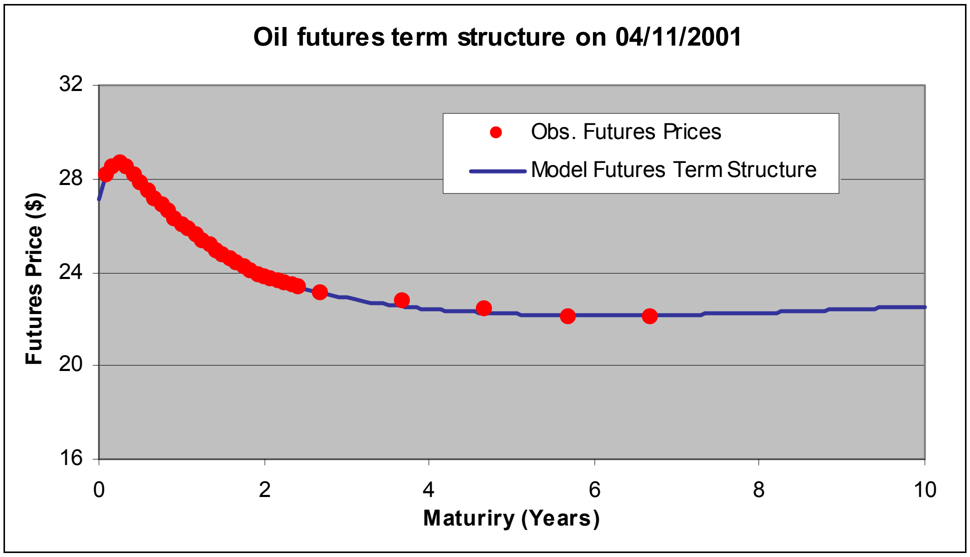

Fig. 3. Estimated and observed oil futures prices on 04/11/2001.

A more quantitative measure of fit to observed futures prices is shown in Table 3, which shows the root mean-square error (RMSE) of model futures estimates for in-sample data in each of the panels. It can be seen that futures price estimations are unbiased and exhibit a low RMSE. For example, for Panel A, considering an average price of spot oil price of $21, the RMSE corresponds to an error of only $0.07.

	RMSE	Bias
Panel A (1992-2001)	0.29%	-0.0001%
Panel B (1992-1996)	0.16%	0.0000%
Panel C (1997-2001)	0.35%	-0.0002%

Table 3. In-Sample RMSE and Bias for Panels A, B and C.

Another measure of model stability can be found by comparing the fit of out-of-sample futures estimations to in-sample ones. For that purpose, we estimate the model using another panel-data (Panel D) which includes futures prices ranging only from 1992 to 2000. Then we compare the RMSE for the year 2001, using the parameters estimated with Panel D with those that were obtained using Panel A. Results are shown in Figure 4, where we plot the RMSE of estimated futures for each maturity. We can see from this figure that the model is very stable even when calculating term structures one year ahead.

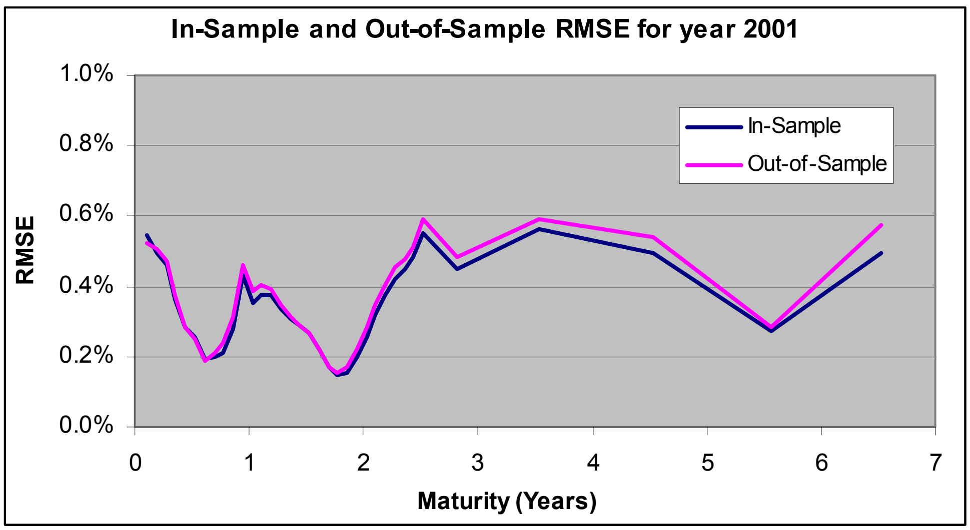

Fig. 4. Estimated and observed oil futures prices on 04/11/2001.

As a last measure of performance, we calculate the model volatility term structure of futures returns and compare it to the empirical volatilities obtained directly from observed futures prices. The empirical volatility of futures returns, for a futures contract maturing in τ years, was calculated as:

$$\hat{\sigma}_F^2(\tau) = \frac{1}{\Delta t}\sum_{i=1}^{N}\left(\log\left(F(t_i,\tau)/F(t_i-\Delta t,\tau)\right)-\bar{\mu}\right)^2 \tag{21}$$

This is shown in Figure 5, where we plot for each maturity the theoretical volatility term structure implied by the model, using parameter estimates obtained with Panel A, and the empirical volatilities obtained directly from futures price data in the same period. We can see that the model closely fits the empirical volatility term structure.

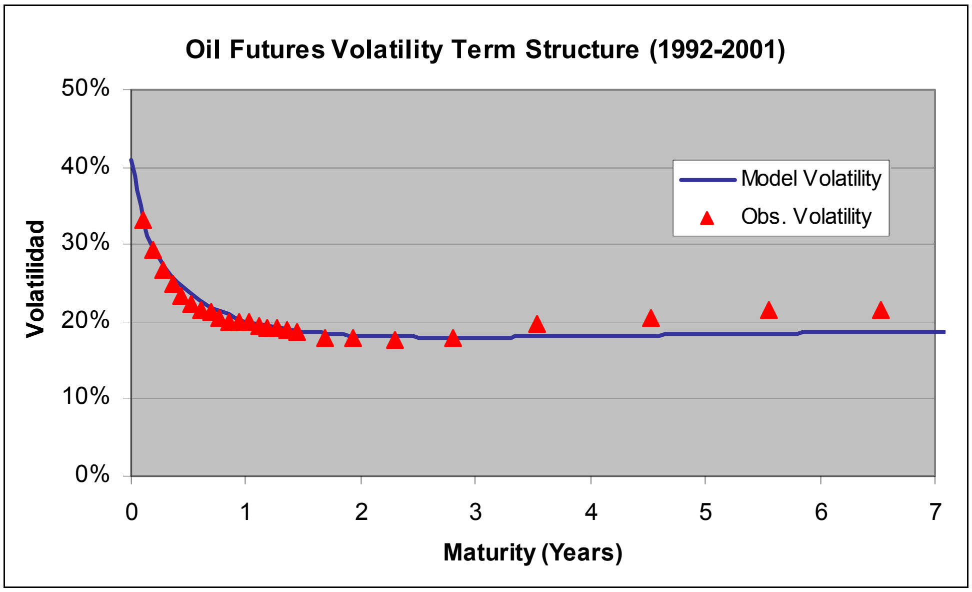

Fig. 5. Volatility Term Structure of Oil Futures Returns 1992 - 2001.

The various analysis performed show that our estimation procedure is able to obtain estimates of a 4-factor lognormal model that accurately fits the first and second moments of observed oil futures prices. This is especially important in the valuation and hedging of long term commodity-linked contingent claims.

5. CONCLUSIONS

The valuation and hedging of commodity contingent-claims is of great importance for financial engineering applications. Producers and consumers of commodities, financial intermediaries and investors, can all benefit from accurate valuation methodologies of commodity-linked financial and real assets, and also from better hedging strategies.

Nevertheless, given the complex behaviour of many commodity markets, which include volatile prices and mean reverting term structures, the application of simple models can lead to unreliable results with vast economic consequences, which may include heavy financial losses to corporations [Culp and Miller (1994)].

The contributions of this paper are twofold. First, we develop a new N-factor model of commodity prices which generalises previous research. Several of the most commonly used commodity price models can be seen as particular cases of this new model [Brennan and Schwartz (1985), Gibson and Schwartz (1990), Ross (1995), Schwartz (1997), Schwartz and Smith (2001) and Cortazar and Schwartz (2003)]. The model is flexible enough not only to yield an accurate valuation of observed futures prices term structures, but also to fit the empirical volatility term structures while maintaining its simplicity and tractability.

Second, we show how to use an incomplete panel-data when applying the Kalman filter. Incomplete panel-data is common in most markets because existing contracts do not trade every day for all maturities and also because new contracts are frequently introduced. Our method does not require us to aggregate or discard any data, a normal procedure used in the literature when applying the Kalman filter which induces great information loss.

We illustrate this procedure by estimating a 4-factor model using all oil futures prices during 10 years. Results show that our model can accurately fit observed futures price term structures and the empirical volatilities structure of futures returns. Model parameters are stable across time and exhibit good out-of-sample properties, a feature that may be critical in real world implementations.

REFERENCES

Babbs, S.H., Nowman, K.B., 1999. Kalman filtering of generalized Vasicek term structure models, Journal of Financial and Quantitative Analysis 34 (1) 115-130.

Ball, C., Torous, W., 1996. Unit roots and the estimation of interest rate dynamics, Journal of Empirical Finance 3 215-238.

Bessembinder, H., Coughenour, J.F., Seguin, P.J., Smoller, M.M., 1995. Mean reversion in equilibrium asset prices: Evidence from the futures term structure, The Journal of Finance 50 (1) 361-375.

Bessembinder, H., Coughenour, J.F., Seguin, P.J., Smoller, M.M., 1996. Is there a term structure of futures volatilities? Reevaluating the Samuelson hypothesis, The Journal of Derivatives 4 (2) 45-58.

Brennan, M.J., 1991. The price of convenience and the valuation of commodity contingent claims. In: Lund, D., Øksendal, B. (Eds.), Stochastic models and option values, Elsevier Science Publishers, North Holland, pp. 33-71.

Brennan, M.J., Schwartz, E.S., 1985. Evaluating natural resources investments, Journal of Business 58 (2) 135-157.

Casassus, J., Collin-Dufresne, P., 2001. 'Maximal' convenience yield model implied by commodity futures, Working Paper, Carnegie Mellon University.

Cortazar, G., Schwartz, E.S., 1994. The valuation of commodity-contingent claims, The Journal of Derivatives 1 27-35.

Cortazar, G., Schwartz, E.S., 1997. Implementing a real option model for valuing an undeveloped oil field, International Transactions in Operational Research 4 (2) 125-137.

Cortazar, G., Schwartz, E.S., 2003. Implementing a stochastic model for oil futures prices, Energy Economics, 25 (3) 215-238.

Cortazar, G., Schwartz, E.S., Naranjo, L., 2003. Term structure estimation in low-frequency transaction markets: A Kalman filter approach with incomplete panel-data, Working Paper, Anderson School of Management, UCLA.

Culp, C.L., Miller, M.H., 1994. Hedging a flow of commodity deliveries with futures: Lessons from Metallgesellschaft, Derivatives Quarterly 1(1) 7-15.

Dai, Q., Singleton, K.J., 2000. Specification analysis of affine term structure models, The Journal of Finance, 55 (5) 1943-1978.

de Jong, F., 2000. Time-series and cross-section information in affine term structure models, Journal of Business & Economics Statistics, 18 (3) 300-314.

de Jong, F., Santa-Clara, P., 1999. The dynamics of the forward interest rate curve: A formulation with state variables, Journal of Financial and Quantitative Analysis, 34 (1) 131-157.

Dewachter, H., Maes, K., 2001. An admissible affine model for joint term structure dynamics of interest rates, CES Discussion Paper Series, Catholic University of Leuven.

Duan, J.-C., Simonato, J.-G., 1999. Estimating and testing exponential-affine term structure models by Kalman filter, Review of Quantitative Finance and Accounting 13 (2), 111-135.

Geyer, A.L.J., Pichler, S., 1999. A state-space approach to estimate and test multifactor Cox-Ingersoll-Ross models of the term structure, The Journal of Financial Research 22 (1) 107-130.

Gibson, R., Schwartz, E.S., 1990. Stochastic convenience yield and the pricing of oil contingent claims, The Journal of Finance 45 (3) 959-976.

Hamilton, J.D., 1994. Time series analysis, Princeton University Press, Princeton, N.J.

Harvey, A.C., 1989. Forecasting, structural time series models and the Kalman filter, Cambridge University Press, Cambridge.

Kalman, R.E., 1960. A new approach lo linear filtering and prediction problems, Journal of Basic Engineering, Transactions of the ASME 82 35-45.

Laughton, D.G., Jacoby, H.D., 1993. Reversion, timing options, and long-term decision-making, Financial Management 22 (3) 225-240.

Laughton, D.G., Jacoby, H.D., 1995. The effects of reversion on commodity projects of different length. In: Trigeorgis, L. (ed.), Real options in capital investments: Models, strategies, and applications, Praeger Publisher, Westport, Conn.

Lund, J., 1994. Econometric analysis of continuous-time arbitrage-free models of the term structure of interest rates, Working Paper, The Aarhus School of Business.

Lund, J., 1997. Non-linear kalman filtering techniques for term-structure models, Working Paper, The Aarhus School of Business.

Øksendal, B., 1998. Stochastic differential equations: an introduction with applications, 5th Edn. Springer, Berlin; New York.

Pennacchi, G.G., 1991. Identifying the dynamics of real interest rates and inflation: Evidence using survey data, The Review of Financial Studies 4 (1) 53-86.

Ross, S.A., 1999. Hedging long run commitments: Exercises in incomplete markets pricing. In: Culp, C., Miller, M. (Eds.), Corporate Hedging in Theory and Practice, Risk Books.

Schwartz, E.S., 1997. The stochastic behavior of commodity prices: Implications for valuation and hedging, The Journal of Finance 52 (3) 923-973.

Schwartz, E.S., Smith, J.E., 2000. Short-term variations and long-term dynamics in commodity prices, Management Science 46 893-911.

Sørensen, C., 2002. Modeling seasonality in agricultural commodity futures, Journal of Futures Markets 22 393-426.

APPENDIX A

In this appendix we determine the equivalence between the 3-factor model of Cortazar and Schwartz (2003) and our new 3-factor model. The model for the spot price S_t in the Cortazar and Schwartz (2003) model is:

$$dS_t = (v_t - y_t)S_t dt + \sigma_1 S_t dw_1 \quad (22)$$

$$dy_t = -\kappa y_t dt + \sigma_2 dw_2 \quad (23)$$

$$dv_t = a(\bar{v} - v_t)dt + \sigma_3 dw_3 \quad (24)$$

where $(dw_1)(dw_2) = \rho_{21}dt$, $(dw_2)(dw_3) = \rho_{32}dt$ and $(dw_1)(dw_3) = \rho_{13}dt$.

This model can be written in logarithmic form:

$$\log S_t = \mathbf{h}'\mathbf{x}_t \quad (25)$$

$$d\mathbf{x}_t = (-\mathbf{K}\mathbf{x}_t + \boldsymbol{\beta})dt + \boldsymbol{\Sigma} d\mathbf{w}_t \quad (26)$$

where $\mathbf{h}' = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}$, $\mathbf{K} = \begin{pmatrix} 0 & 1 & -1 \\ 0 & \kappa & 0 \\ 0 & 0 & a \end{pmatrix}$, $\boldsymbol{\beta} = \begin{pmatrix} -1/2\,\sigma_1^2 \\ 0 \\ a\bar{v} \end{pmatrix}$, $\boldsymbol{\Sigma} = \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{pmatrix}$,

$\mathbf{x} = \begin{pmatrix} x_t \\ y_t \\ v_t \end{pmatrix}$ and $d\mathbf{w}_t d\mathbf{w}_t' = \boldsymbol{\Omega} dt = \begin{pmatrix} 1 & \rho_{21} & \rho_{31} \\ \rho_{21} & 1 & \rho_{32} \\ \rho_{31} & \rho_{32} & 1 \end{pmatrix} dt$.

To obtain the relationship between the two models, we apply the following affine transformation over the original state variable vector $\mathbf{x}_t$:

$$\boldsymbol{\zeta}_t = \mathbf{L}\mathbf{x}_t + \boldsymbol{\varphi} \quad (27)$$

where $\mathbf{L} = \begin{pmatrix} 1 & -1/\kappa & 1/a \\ 0 & 1/\kappa & 0 \\ 0 & 0 & -1/a \end{pmatrix}$ and $\boldsymbol{\varphi} = \begin{pmatrix} -\bar{v}/a - (\bar{v} - 1/2\sigma_1^2)t \\ 0 \\ \bar{v}/a \end{pmatrix}$.

The new vector $\boldsymbol{\zeta}_t$ corresponds to the state variables in our new model. Note that this transformation is invertible, and therefore it establishes a one-to-one correspondence between the parameters and state variables of the two models. This way, we obtain that:

$$\log S_t = \mathbf{1}'\boldsymbol{\zeta}_t + \left(\overline{\nu} - \frac{1}{2}\sigma_1^2\right)t \tag{28}$$

$$d\boldsymbol{\zeta}_t = -\overline{\mathbf{K}}\boldsymbol{\zeta}_t dt + \overline{\boldsymbol{\Sigma}} d\overline{\mathbf{w}}_t \tag{29}$$

where

$$\overline{\mathbf{K}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \kappa & 0 \\ 0 & 0 & a \end{pmatrix}, \ \overline{\boldsymbol{\Sigma}} = \begin{pmatrix} \overline{\sigma}_1 & 0 & 0 \\ 0 & \overline{\sigma}_2 & 0 \\ 0 & 0 & \overline{\sigma}_3 \end{pmatrix} \text{ and } d\overline{\mathbf{w}}_t d\overline{\mathbf{w}}_t' = \overline{\boldsymbol{\Omega}} dt = \begin{pmatrix} 1 & \overline{\rho}_{21} & \overline{\rho}_{31} \\ \overline{\rho}_{21} & 1 & \overline{\rho}_{32} \\ \overline{\rho}_{31} & \overline{\rho}_{32} & 1 \end{pmatrix} dt .$$

The eigenvalues of mean reversion matrices $\mathbf{K}$ and $\overline{\mathbf{K}}$ are the same.

The relationship existing between the variance-covariance parameters of the two models is given by the following equation:

$$\overline{\boldsymbol{\Sigma}}\overline{\boldsymbol{\Omega}}\overline{\boldsymbol{\Sigma}} = \mathbf{L}\boldsymbol{\Sigma}\boldsymbol{\Omega}\boldsymbol{\Sigma}\mathbf{L}' \tag{30}$$

Assuming constant market prices of risk, the risk adjusted process of Cortazar and Schwartz (2003) is:

$$\log S_t = \mathbf{h}'\mathbf{x}_t \tag{31}$$

$$d\mathbf{x}_t = (-\mathbf{K}\mathbf{x}_t + \boldsymbol{\beta} - \boldsymbol{\lambda})dt + \boldsymbol{\Sigma} d\mathbf{w}_t \tag{32}$$

The risk adjusted process of our new model is:

$$\log S_t = \mathbf{1}'\boldsymbol{\zeta}_t + \left(\overline{\nu} - \frac{1}{2}\sigma_1^2\right)t \tag{33}$$

$$d\boldsymbol{\zeta}_t = (-\overline{\mathbf{K}}\boldsymbol{\zeta}_t - \overline{\boldsymbol{\lambda}})dt + \overline{\boldsymbol{\Sigma}} d\overline{\mathbf{w}}_t \tag{34}$$

Then, the following relationship between the market prices of risk of the two models must hold:

$$\overline{\boldsymbol{\lambda}} = \mathbf{L}\boldsymbol{\lambda} \tag{35}$$

APPENDIX B

In this appendix we describe in detail how to apply the Kalman filter estimation procedure developed in Section 3 to the generalized log-normal model introduced in Section 2, with an incomplete panel-data set of futures prices .

The transition equation of the state variables under a generalized log-normal model is independent of the observations, and the associated terms appearing in equation (2) are:

$$\mathbf{A}_t = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1-k_2\Delta t & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1-k_n\Delta t \end{pmatrix} \qquad \mathbf{c}_t = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \tag{36}$$

$$\mathbf{Q}_t = \begin{pmatrix} \sigma_1^2 & \cdots & \sigma_1\sigma_n\rho_{1n} \\ \vdots & \ddots & \vdots \\ \sigma_n\sigma_1\rho_{n1} & \cdots & \sigma_n^2 \end{pmatrix} \Delta t \tag{37}$$

where Δt is the time interval at which futures prices are observed, and other parameters are the ones appearing in equation (2).

Let m_t be the number at time t of observed futures prices, $\left\{F_{i,t}\right\}_{i=1}^{m_t}$ the set of observed futures prices at time t and $\left\{\tau_{i,t}\right\}_{i=1}^{m_t}$ the set containing their respective associated maturities. The vector containing the logarithm of futures prices will be denoted by $\mathbf{z}_t$, a $m_t \times 1$ vector:

$$\mathbf{z}_t = \begin{pmatrix} \log F_{1,t} \\ \vdots \\ \log F_{m_t,t} \end{pmatrix} \tag{38}$$

The parameters of the measurement equation are:

$$\mathbf{H}_t = \begin{pmatrix} 1 & \mathrm{e}^{-\kappa_2\tau_{1,t}} & \cdots & \mathrm{e}^{-\kappa_n\tau_{1,t}} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \mathrm{e}^{-\kappa_2\tau_{m_t,t}} & \cdots & \mathrm{e}^{-\kappa_n\tau_{m_t,t}} \end{pmatrix} \tag{39}$$

$$\mathbf{d}_t = \begin{pmatrix} v_{1,t} \\ \vdots \\ v_{m_t,t} \end{pmatrix} \tag{40}$$

where

$$v_{i,t} = \mu t + (\mu - \lambda_1 + \frac{1}{2}\sigma_1^2)\tau_{k,t} - \sum_{i=2}^{N} \frac{1-\mathrm{e}^{-\kappa_i \tau_{k,t}}}{\kappa_i}\lambda_i + \frac{1}{2}\sum_{i,j\neq 1} \sigma_i \sigma_j \rho_{ij} \frac{1-e^{-(\kappa_i+\kappa_j)\tau_{k,t}}}{\kappa_i + \kappa_j} \tag{41}$$

The remaining parameters to be specified belong to the covariance matrix of measurement errors $\mathbf{R}_t$. In this paper, we assume that this covariance matrix is diagonal that can only have 1 parameter ξ representing the variance of measurement errors of futures prices which is assumed to be the same for all futures contracts:

$$\mathbf{R}_t = \begin{pmatrix} \xi & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \xi \end{pmatrix} \tag{42}$$

Of course, this assumption can be easily relaxed.